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On the symmetry of simple 12- and 13-hedra

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The symmetry point groups for all combinatorially non-isomorphic simple 12- and 13-hedra (7595 and 49566, respectively) are contributed in the paper for the first time. The most symmetrical polyhedra with automorphism group orders not less than 3 are drawn as Schlegel diagrams and characterized by the facet symbols and symmetry point groups.

1. Introduction

The symmetry point groups and Schlegel diagrams for all combinatorially non-isomorphic 4- to 8-hedra and simple (only 3 edges meet at each vertex) 9- to 11-hedra were contributed in our previous papers (Voytekhovsky, 1999, 2000, 2001*a*). The symmetry point groups for all combinatorially non-isomorphic non-simple 9- and 10-hedra with Schlegel diagrams of the most symmetrical (with automorphism group orders not less than 3) shapes were given in the paper by Voytekhovsky & Stepenshchikov (2002). Here, we report the symmetry point groups for the simple 12-hedra (7595 in total), the automorphism group orders of which were earlier found by Duijvestijn & Federico (1981). The symmetry point groups for the simple 13-hedra are also reported here for the first time. Their number (49566) was given first in Engel (1994).

2. Generation and characterization of polyhedra

As in our previous papers, we generate polyhedra as their Schlegel diagrams. This method is obviously justified by two theorems: (i) every 3-connected planar graph can be realized as a polyhedron, and (ii) every combinatorial automorphism of a polyhedron is affinely realizable. That is, there exists to each Schlegel diagram a three-dimensional polyhedral realization such that its edge graph is isomorphic to the diagram while its symmetry point group is isomorphic to the automorphism group of the diagram.

The polyhedra were generated by the Fedorov (1893) recurrence algorithm briefly described by Engel (1994) and Voytekhovsky (2001*b*). As the simple 11-hedra have previously been found, we used them to generate the simple 12- and, afterwards, 13-hedra. Besides, it has already been found (Voytekhovsky, 2001*b*) that there is only one simple 12-hedron and no simple 13-hedra without triangular and quadrilateral facets simultaneously. Hence, all we need is to apply α and β procedures of the Fedorov algorithm, which are intended to cut any vertex and edge of a simple polyhedron to generate new triangular and quadrilateral facets, respectively. After generated polyhedra have been compared in Schlegel diagrams, each combinatorially non-isomorphic shape was characterized by a symmetry point group and a facet symbol. The latter shows the numbers $[n_3 n_4 \ldots n_{\max}]$ of triangular, quadrilateral *etc.* facets of the polyhedron in a sequence.

3. Results and discussion

The automorphism group order statistics of simple 12-hedra are as follows: 1–6756, 2–747, 3–1, 4–68, 6–7, 8–10, 10–1, 12–3, 40–1, 120–1. Two of the above automorphism group orders can be easily identified as 1 and 3 symmetry point groups. But others were checked by us with the following symmetry point-group statistics: 1–6756, *m*–597, 2–146, $\bar{1}$–4, 3–1, *mm*2–53, 2/*m*–10, 222–3, $\bar{4}$–2, 3*m*–5, 32–1, $\bar{6}$–1, $\bar{4}2m$–6, *mmm*–4, 5*m*–1, $\bar{3}m$–2, $\bar{6}m2$–1, 10/*mmm*–1 (prism), and $\bar{3}\bar{5}m$–1 (dodecahedron). This result was previously announced in Voytekhovsky (2001*a*). We confirm it here.

The automorphism group order statistics of simple 13-hedra are as follows: 1–47030, 2–2377, 3–12, 4–118, 6–28, 44–1. The corresponding symmetry point-group statistics is as follows: 1–47030, *m*–1952, 2–425, 3–12, *mm*2 –118, 3*m*–28, $\bar{2}\bar{2}m2$–1 (prism). Both of them are contributed here for the first time.

The most symmetrical simple 12-hedra with automorphism group orders not less than 3 are given in Fig. 1 with their Schlegel diagrams. Their symmetry point groups are: $\bar{3}\bar{5}m$: No. 1; 10/*mmm*: 2; $\bar{6}m2$: 4; $\bar{3}m$: 15, 88; 5*m*: 87; *mmm*: 6, 14, 21, 79; $\bar{4}2m$: 7–9, 26, 64, 85; $\bar{6}$: 58; 32: 13; 3*m*: 19, 59, 61–63; $\bar{4}$: 67, 84; 222: 5, 66, 68; 2/*m*: 11, 32, 41, 42, 46, 50, 65, 69, 70, 76; *mm*2: 3, 10, 12, 16–18, 20, 22–25, 27–31, 33–40, 43–45, 47–49, 51–57, 71–75, 77, 78, 80–83, 86, 89–92; 3: 60.

Their facet symbols are: 1 – [0,0,12], 2 – [0,10,0,0,0,0,0,2], 3 – [0282], 4 – [0363], 5–9 – [0444], 10, 11 – [04602], 12 – [046101], 13–15 – [0606], 16–18 – [06222], 19 – [06303], 20, 21 – [064002], 22 – [07032], 23, 24 – [070401], 25 – [072102], 26 – [08004], 27, 28 – [080202], 29, 30 – [0820002], 31, 32 – [2064], 33 – [20802], 34, 35 – [2145], 36 – [216201], 37, 38 – [2226], 39 – [22422], 40–42 – [24042], 43, 44 – [240501], 45, 46 – [24204], 47, 48 – [242202], 49, 50 – [2440002], 51 – [24400101], 52 – [25014], 53, 54 – [260022], 55 – [2602002], 56 – [26200002], 57 – [280000002], 58 – [3036], 59 – [30603], 60, 61 – [33033], 62, 63 – [333003], 64 – [4008], 65 – [40242], 66, 67 – [40404], 68–70 – [404202], 71 – [40430001], 72 – [41240001], 73–76 – [42024], 77–79 – [420402], 80 – [422022], 81 – [4222002], 82 – [430041], 83 – [430203], 84, 85 – [440004], 86 – [44020002], 87 – [50150001], 88 – [60006], 89 – [600141], 90 – [600222], 91 – [6004002], 92 – [602004].

The most symmetrical simple 13-hedra with automorphism group orders not less than 3 are given in Fig. 2 with their Schlegel diagrams. Their symmetry point groups are: $\bar{2}\bar{2}m2$: No. 2; 3*m*: 5, 6, 20–22, 42, 46–49, 51, 98, 99, 102, 106, 107, 113, 114, 123, 124, 136, 140, 141, 146, 152, 154, 155, 159; *mm*2: 1, 3, 4, 7–19, 23–41, 43–45, 52–95, 108–111, 115–122, 125–134, 137, 138, 142–145, 147–151, 156–158; 3: 50, 96, 97, 100, 101, 103–105, 112, 135, 139, 153.

Their facet symbols are: 1 – [0,1,10,2], 2 – [0,11,0,0,0,0,0,0,2], 3–6 – [0364], 7 – [03802], 8, 9 – [0445], 10 – [04612], 11–14 – [046201], 15, 16 – [0526], 17 – [056002], 18 – [0607], 19 – [06232], 20, 21 – [06313], 22 – [0633001], 23 – [064021], 24 – [064102], 25, 26 – [07042], 27 – [07204], 28–31 – [072202], 32, 33 – [0740002], 34, 35 – [080221], 36 – [080302], 37 – [08040001], 38 – [08202001], 39 – [0821002], 40, 41 – [090022], 42 – [0900301], 43 – [0902002], 44, 45 – [09200002], 46 – [1093], 47, 48 – [1336], 49 – [13603], 50, 51 – [16033], 52–54 – [21622], 55 – [218002], 56–61 – [2227], 62–64 – [22432], 65–67 – [224401], 68 – [226021], 69 – [226102], 70, 71 – [2308], 72 – [23242], 73 – [23404], 74 – [234202], 75 – [2360002], 76 – [24052], 77, 78 – [240601], 79, 80 – [24214], 81 – [242302], 82 – [25024], 83–85 – [250402], 86 – [252022], 87 – [25400002], 88, 89 – [260203], 90 – [2603002], 91 – [26210002], 92 – [270004], 93, 94 – [27020002], 95 – [2900000002], 96 – [3037], 97 – [3063001], 98, 99 – [33043], 100 – [3330301], 101, 102 – [333103], 103 – [3333000001], 104 – [3600031], 105 – [3600300001], 106, 107 – [3601003], 108, 109 – [40252], 110, 111 – [402601], 112–114 – [40333], 115–117 – [40414], 118 – [404221], 119–121 – [404302], 122–124 – [406003], 125 – [40602001], 126, 127 – [41062], 128 – [41224], 129 – [412402], 130, 131 – [42034], 132 – [422041], 133 – [422203], 134 – [42400201], 135, 136 – [43006], 137, 138 – [430222], 139–141 – [430303], 142, 143 – [4304002], 144 – [432004], 145 – [43220002], 146 – [4330003], 147 – [440023], 148 – [4402012], 149 – [4402200001], 150 – [4420020001], 151, 152 – [600403], 153 – [6030031], 154 – [6030300001], 155 – [6031003], 156 – [610204], 157 – [61040002], 158 – [62020102], 159 – [700033].

As for the cases of 4- to 10- and simple 11-hedra, the shapes of 1, *m*, 2 and *mm*2 symmetry point groups also prevail among the simple 12- and 13-hedra. This is in conformity with our idea that this tendency is a general property of the abstract polyhedra variety (Voytekhovsky & Stepenshchikov, 2002). The number of polyhedra rapidly drops with growing symmetry with the trivial (of 1 symmetry point group) shapes forming the overwhelming majority.

4. Conclusions

Up to now, the whole combinatorial variety of 4- to 10- and simple 11- to 13-hedra have been generated and characterized by facet symbols and symmetry point groups. The most symmetrical shapes (usually

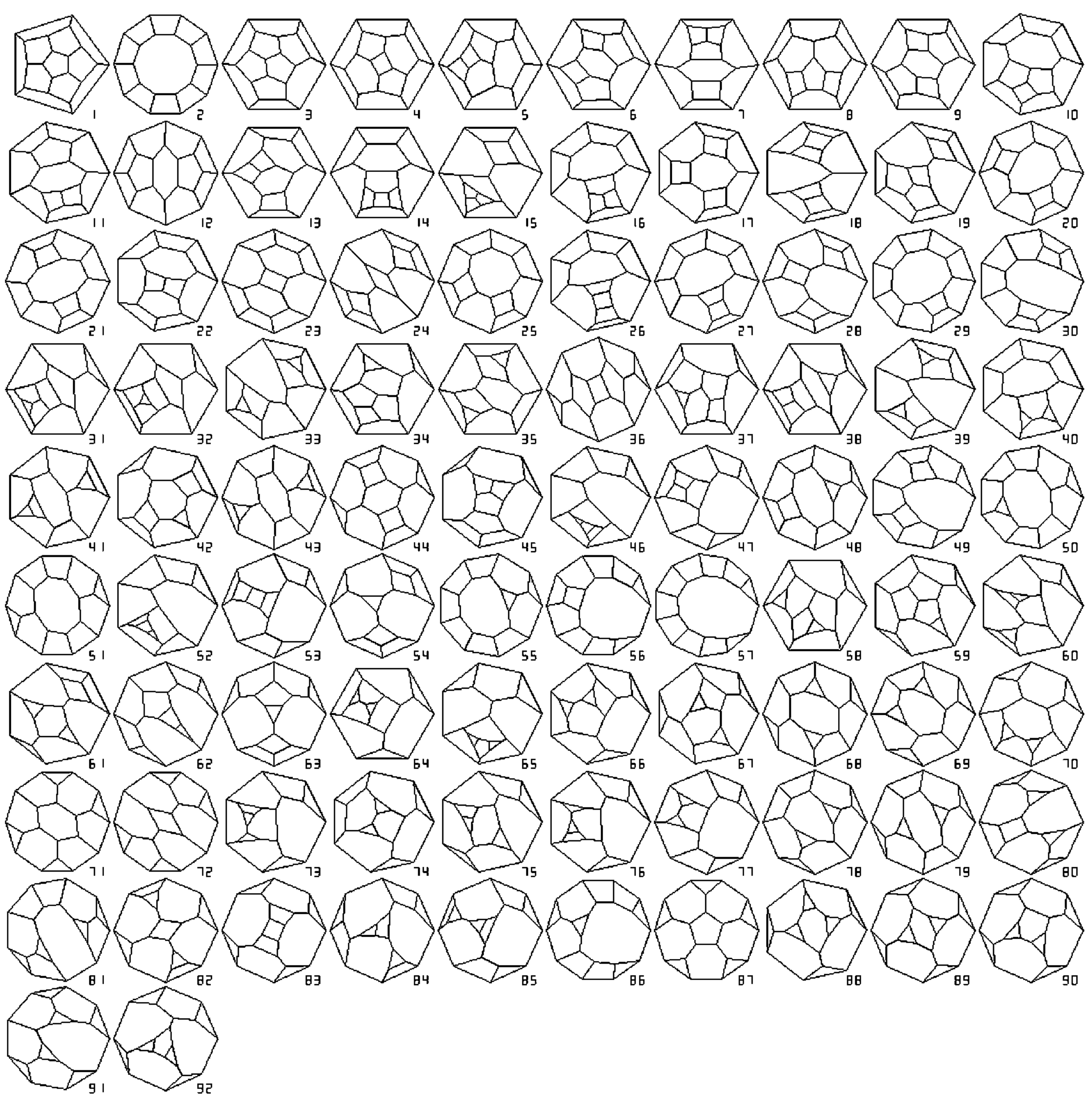

Figure 1
The most symmetrical simple 12-hedra. See text for description.

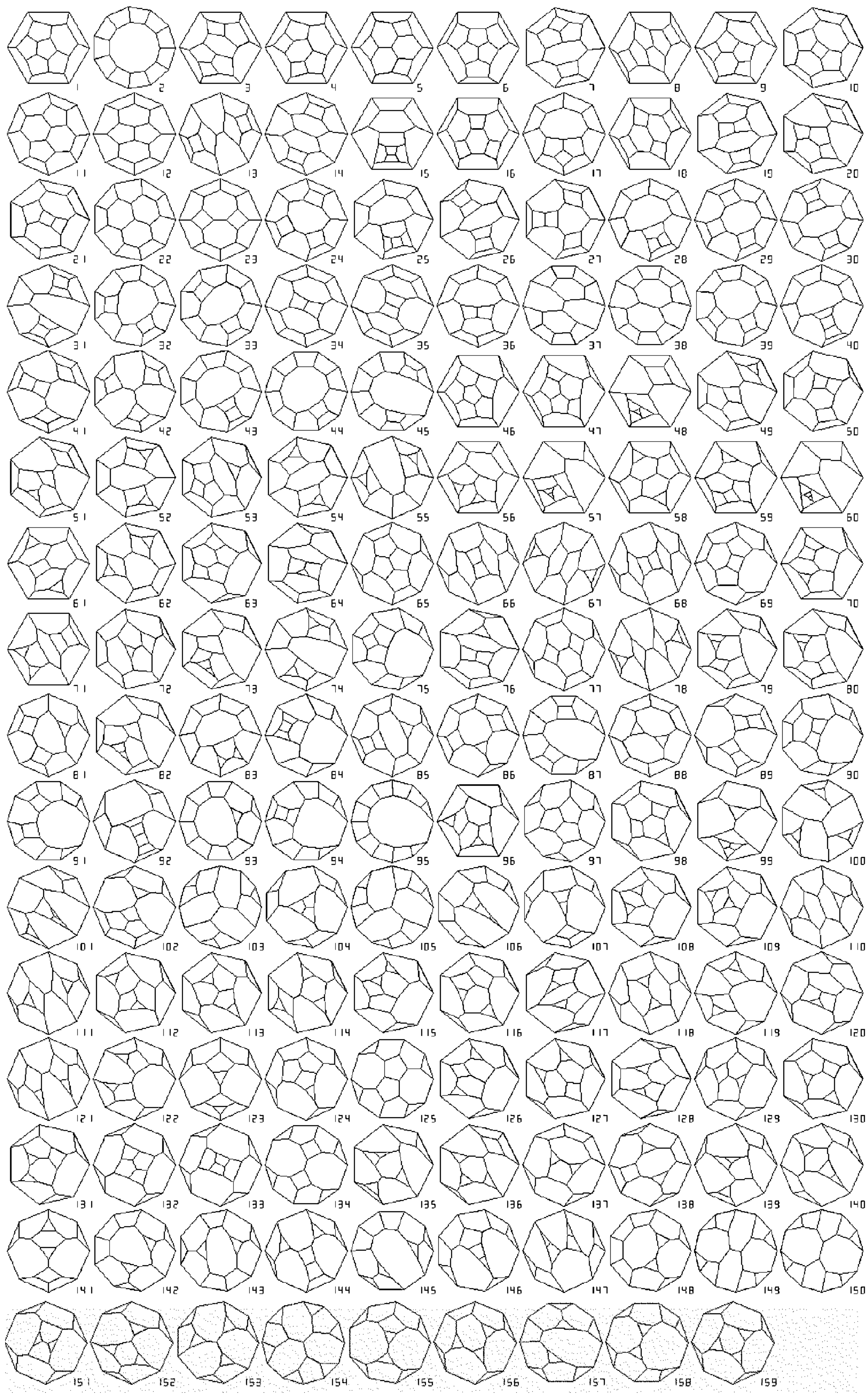

Figure 2
The most symmetrical simple 13-hedra. The diagrams Nos. 60, 99 and 101 can be clearly deciphered taking into account their simplicity and facet symbols given in the text.

those with automorphism group orders not less than 3) are drawn as Schlegel diagrams. Our next steps are to generate and characterize all the non-simple 11- and simple 14-hedra. The overwhelming majority of simple as well as non-simple polyhedra belongs to the trivial symmetry point group. Hence, one more problem is to find some methods to classify them.

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